Quiz 5 Solution. Linear Algebra basics.

Calculus 4. May 20nd, 2021

1. Find the value of α such that the following vectors are linearly dependent.

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ \alpha \\ 10 \end{bmatrix}$$

Solution:

Method 1: By definition.

The three vectors are linearly dependent if there exist a nonzero vector (x, y, z) such that

$$x \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} + z \begin{bmatrix} 5 \\ \alpha \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence we have the system

$$2x + y + 5z = 0, \quad 3y + \alpha z = 0, \quad x + 3y + 10z = 0$$

and we want to find the α value that would lead to a nonzero solution.

If $z = 0$, then $y = 0$ and $x = 0$ follows. So we now assume $z \neq 0$.

By plugging $x = -3y - 10z$ into the first equation, we get $-5y - 15z = 0$, or $y = -3z$. Hence the second equation is now $(\alpha - 9)z = 0$, so $\alpha = 9$. $\square$

Method 2: By determinant equal to zero.

Linearly dependent would imply that the 3×3 matrix $\begin{bmatrix} 2 & 1 & 5 \\ 0 & 3 & \alpha \\ 1 & 3 & 10 \end{bmatrix}$ would have zero as its determinant.

$$\det \begin{bmatrix} 2 & 1 & 5 \\ 0 & 3 & \alpha \\ 1 & 3 & 10 \end{bmatrix} = 60 + \alpha - 15 - 6\alpha = 45 - 5\alpha = 0 \quad \Rightarrow \quad \alpha = 9$$

$\square$

Method 3: By Gaussian elimination.

We apply the Gaussian elimination to the matrix $\begin{bmatrix} 2 & 0 & 1 \\ 1 & 3 & 3 \\ 5 & \alpha & 10 \end{bmatrix}$. Linearly dependent would mean the resulting row-echelon form would have at least one row of zeros.

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 3 & 3 \\ 5 & \alpha & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0.5 \\ 1 & 3 & 3 \\ 5 & \alpha & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 3 & 2.5 \\ 0 & \alpha & 7.5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & \frac{5}{6} \\ 0 & \alpha & 7.5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & \frac{5}{6} \\ 0 & 0 & 7.5 - \frac{5\alpha}{6} \end{bmatrix}$$

Hence $\frac{15}{2} - \frac{5\alpha}{6} = 0$, implying $\alpha = 9$. $\square$

2. Find the rank of the matrices A and B. State your method.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ -2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & -1 & 4 & 3 \\ 1 & -2 & -1 & 3 \end{bmatrix}$$

Solution:

Method 1: By Gaussian elimination.

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ -2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & -5 \\ 0 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Rank of A is 2.

$$\begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & -1 & 4 & 3 \\ 1 & -2 & -1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & -1 & 4 & 3 \\ 0 & 1 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 3 & 6 & -3 \\ 0 & 1 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of B is 2. $\square$

Method 2: Dimension of row space.

Same as above. $\square$

Method 3: Dimension of column space.

Gaussian elimination after transpose.

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 \\ 0 & -5 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1.4 \end{bmatrix}$$

Rank of A is 2.

$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & -2 \\ 2 & 4 & -1 \\ -1 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 2 & 4 & -1 \\ -1 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 6 & 3 \\ 0 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank of B is 2. $\square$

3. Compute the determinant of the matrix

$$M = \begin{bmatrix} 0 & 0 & -2 & 6 \\ 5 & 0 & 0 & 3 \\ 0 & 5 & -1 & 2 \\ 3 & 2 & 0 & 0 \end{bmatrix}$$

Solution:

$$\det \begin{bmatrix} 0 & 0 & -2 & 6 \\ 5 & 0 & 0 & 3 \\ 0 & 5 & -1 & 2 \\ 3 & 2 & 0 & 0 \end{bmatrix} = 0 - 0 + (-2)\det \begin{bmatrix} 5 & 0 & 3 \\ 0 & 5 & 2 \\ 3 & 2 & 0 \end{bmatrix} - 6\det \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & -1 \\ 3 & 2 & 0 \end{bmatrix} = 90 + 40 - 60 = 70$$

□

4. Find all eigenvalues of the matrix

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$$

Solution:

$$\det \begin{bmatrix} x-5 & -2 \\ -2 & x-5 \end{bmatrix} = x^2 - 10x + 25 - 4 = x^2 - 10x + 21 = (x-3)(x-7)$$

Eigenvalues are 3 and 7. □

5. The matrix $A = \begin{bmatrix} -1 & 2 & 5 \\ 2 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ has an eigenvalue $\lambda = 2$. Find a corresponding eigenvector $\mathbf{v}$.

Solution:

$$\begin{bmatrix} -1 & 2 & 5 \\ 2 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} -3 & 2 & 5 \\ 2 & -3 & 0 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Use Gaussian elimination

$$\begin{bmatrix} -3 & 2 & 5 \\ 2 & -3 & 0 \\ 1 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 2 & -3 & 0 \\ -3 & 2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & -3 & 6 \\ 0 & 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

We get $x = 3z$ and $y = 2z$, therefore an eigenvector for $\lambda = 2$ is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ □

Grading scheme: 4 points for each problem. Because all five problems are computational, we give 2 points for each correct approach and the other points for correct computations. No points will be given for correct answers that don't have all the work leading to it.